

Thermal Correlators from Rindler-AdS₂/CFT₁*

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Abstract

In this paper we study one-dimensional conformal field theory at finite temperature dual to the two-dimensional anti-de Sitter spacetime in the Rindler coordinates. We show that conformal symmetry for thermal two-point functions manifests itself in a form of recurrence relations in the complex frequency space. It is discussed that all the real-time two-point functions are given by solutions to the recurrence relations.

1 Introduction

The purpose of this paper is to report a simple Lie algebraic approach to momentum-space two-point functions of finite-temperature conformal field theory initiated in Refs. [1, 2]. For the sake of simplicity, in this paper we would like to focus on frequency-space thermal two-point functions for a scalar primary operator of one-dimensional conformal field theory (CFT₁). We shall show that conformal symmetry for thermal two-point functions $G_\Delta(\omega)$ in frequency space manifests itself in the form of linear functional relations (or *recurrence relations* in the complex ω -plane)

$$G_\Delta(\omega) = \frac{-1 + \Delta \pm \frac{i\omega}{2\pi T}}{-\Delta \pm \frac{i\omega}{2\pi T}} G_\Delta(\omega \pm i2\pi T), \quad (1)$$

where Δ is the scaling dimension of scalar primary operator and T is the temperature. All the real-time two-point functions are turned out to satisfy these recurrence relations. To derive Eq. (1), we utilize the AdS/CFT correspondence and consider the two-dimensional anti-de Sitter spacetime in the Rindler coordinates (Rindler-AdS₂). The Rindler-AdS₂ (*aka* the AdS₂ black hole [3]) is just a portion of AdS₂ where the time-translation Killing vector generates the noncompact Lorentz group $SO(1, 1)$; see Table 1. In an appropriate parameterization such an AdS₂ portion can be described by the following black-hole-like metric:

$$ds_{\text{Rindler-AdS}_2}^2 = -\left(\frac{r^2}{\ell^2} - 1\right) dt^2 + \frac{dr^2}{r^2/\ell^2 - 1}, \quad r \in (\ell, \infty), \quad (2)$$

where $r = \infty$ corresponds to the AdS₂ boundary while $r = \ell$ the Rindler horizon with ℓ being the AdS₂ radius. In contrast to the well-known global and Poincaré coordinate patches, the Rindler-AdS₂ becomes thermal thanks to the $SO(1, 1)$ time-translation group and is known to be dual to a finite-temperature CFT₁ with nonzero temperature $T = 1/(2\pi\ell)$.¹ The goal of this paper is to show that, with the aid of the real-time prescription of the AdS/CFT correspondence [6, 7], the recurrence relations

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¹Thermal aspects of Rindler-AdS have also been studied in [4, 5].

coordinate patch	time-translation group		frequency spectrum	
	Lorentzian	Euclidean	Lorentzian	Euclidean
global	$SO(2)$	$SO(1,1)$	discrete	continuous
Poincaré	$E(1)$	$E(1)$	continuous	continuous
Rindler	$SO(1,1)$	$SO(2)$	continuous	discrete

Table 1: Three distinct AdS_2 coordinate patches. The time-translation Killing vectors in the global, Poincaré and Rindler coordinates in Lorentzian signature generate the one-parameter subgroups $SO(2)$, $E(1)$, $SO(1,1) \subset SO(2,1)$, respectively. It can be understood from the following that the Rindler- AdS_2 becomes thermal: i) the noncompact Lorentz group $SO(1,1)$ becomes the compact rotation group $SO(2)$ under the Wick rotation; ii) the spectrum of compact $SO(2)$ is always discretized; and iii) discretized frequencies in Euclidean signature are nothing but the Matsubara frequencies in finite-temperature field theory.

(1) just follow from the $SO(2,1)$ isometry of Rindler- AdS_2 . To see this, it is convenient to introduce a new spatial coordinate x via

$$r = \ell \coth(x/\ell), \quad x \in (0, \infty), \quad (3)$$

where $x = 0$ corresponds to the AdS_2 boundary while $x = \infty$ the Rindler horizon. It is easy to check that in this new coordinate system (t, x) the metric becomes conformally flat:

$$ds_{\text{Rindler-}AdS_2}^2 = \frac{-dt^2 + dx^2}{\sinh^2(x/\ell)}. \quad (4)$$

In the following we shall analyze a finite-temperature CFT_1 residing on the boundary $x = 0$.

The rest of the paper is organized as follows. In Section 2 we first briefly discuss unitary representations of the Lie algebra $\mathfrak{so}(2,1)$ in the basis in which the $SO(1,1)$ generator becomes diagonal and then introduce a coordinate realization of $SO(2,1)$ generators which are given by the Killing vectors of Rindler- AdS_2 . We then derive the recurrence relations (1) for frequency-space two-point functions in Section 3. We shall see that two-point Wightman functions and advanced/retarded two-point functions are obtained as “minimal” solutions of Eq. (1).

In the rest of the paper we will work in the units $\ell = 1$ (i.e., $2\pi T = 1$).

2 UIR of $SO(2,1)$ in the $SO(1,1)$ diagonal basis

Let us start with the Lie algebra $\mathfrak{so}(2,1)$ of the one-dimensional conformal group $SO(2,1)$, which is spanned by the three self-adjoint generators $\{J_1, J_2, J_3\}$ satisfying the commutation relations

$$[J^1, J^2] = +iJ^3, \quad [J^2, J^3] = -iJ^1, \quad [J^3, J^1] = -iJ^2. \quad (5)$$

Note that J^3 generates the compact rotation group $SO(2)$, whereas J^1 and J^2 generate the noncompact Lorentz group $SO(1,1)$. We are interested in Unitary Irreducible Representations (UIRs) of $SO(2,1)$ in the basis in which the $SO(1,1)$ generator becomes diagonal, because, as mentioned in Section 1, in the Rindler- AdS_2 the time-translation Killing vector generates the noncompact subgroup $SO(1,1) \subset SO(2,1)$. In order to study the UIRs in the $SO(1,1)$ diagonal basis, we introduce the following *hermitian* linear combinations:

$$J^\pm := J^2 \pm J^3, \quad (6)$$

which satisfy the commutation relations

$$[J^1, J^\pm] = \pm iJ^\pm, \quad [J^+, J^-] = 2iJ^1. \quad (7)$$

The quadratic Casimir C of the Lie algebra $\mathfrak{so}(2, 1)$ is given by

$$C = -(J^1)^2 - (J^2)^2 + (J^3)^2 = -J^1(J^1 \pm i) - J^\mp J^\pm, \quad (8)$$

which commutes with all the generators, $[C, J^a] = 0$ ($a = 1, 2, 3$), as it should. Let $|\Delta, \omega\rangle$ be a simultaneous eigenstate of the Casimir operator C and the $SO(1, 1)$ generator J^1 that satisfy the eigenvalue equations

$$C|\Delta, \omega\rangle = \Delta(\Delta - 1)|\Delta, \omega\rangle, \quad (9a)$$

$$J^1|\Delta, \omega\rangle = \omega|\Delta, \omega\rangle. \quad (9b)$$

It then follows from the commutation relations $[J^1, J^\pm] = \pm i J^\pm$ that the states $J^\pm|\Delta, \omega\rangle$ satisfy the relations $J^1 J^\pm|\Delta, \omega\rangle = ([J^1, J^\pm] + J^\pm J^1)|\Delta, \omega\rangle = (\omega \pm i)J^\pm|\Delta, \omega\rangle$, which imply that J^\pm raises and lowers the eigenvalue ω by $\pm i$:

$$J^\pm|\Delta, \omega\rangle \propto |\Delta, \omega \pm i\rangle. \quad (10)$$

Let us now turn to the Rindler-AdS₂ problem. In the Rindler coordinates the Killing vectors of AdS₂ are turned out to be given by the following first-order differential operators:

$$J^1 = i\partial_t, \quad (11a)$$

$$J^\pm = e^{\pm t} [\sinh x (i\partial_x) \pm \cosh x (i\partial_t)], \quad (11b)$$

which indeed satisfy the commutation relations (7) and are (formally) self-adjoint with respect to the integration measure $\sqrt{|g|}d^2x = dt dx / \sinh^2 x$. A straightforward calculation shows that the quadratic Casimir just gives the d'Alembertian $\square = (1/\sqrt{|g|})\partial_\mu \sqrt{|g|}g^{\mu\nu}\partial_\nu$ on the Rindler-AdS₂

$$C = \sinh^2 x (-\partial_t^2 + \partial_x^2) = \square. \quad (12)$$

The eigenvalue equations (9a) and (9b) then reduce to the following differential equations:

$$i\partial_t \Phi_{\Delta, \omega}(t, x) = \omega \Phi_{\Delta, \omega}(t, x), \quad (13a)$$

$$\left(-\partial_x^2 + \frac{\Delta(\Delta - 1)}{\sinh^2 x}\right) \Phi_{\Delta, \omega}(t, x) = \omega^2 \Phi_{\Delta, \omega}(t, x). \quad (13b)$$

Notice that Eq. (13b) is nothing but the Klein-Gordon equation $(\square - m^2)\Phi_{\Delta, \omega} = 0$ for a scalar field of definite frequency, $\Phi_{\Delta, \omega}(t, x) = e^{-i\omega t} \Phi_{\Delta, \omega}(x)$, where the bulk mass m^2 and the scaling dimension Δ of dual CFT₁ operator are related as $\Delta(\Delta - 1) = m^2$.

3 Thermal two-point functions

A finite-temperature CFT₁ dual to the Rindler-AdS₂ lives on the boundary $x = 0$. To analyze this, let us consider the following asymptotic near-boundary behaviors of Killing vectors:

$$J_{\text{near}}^1 := \lim_{x \rightarrow 0} J^1 = i\partial_t, \quad (14a)$$

$$J_{\text{near}}^\pm := \lim_{x \rightarrow 0} J^\pm = e^{\pm t} (ix\partial_x \pm i\partial_t), \quad (14b)$$

which of course satisfy the commutation relations (7). The near-boundary quadratic Casimir $C_{\text{near}} = -J_{\text{near}}^1(J_{\text{near}}^1 \pm i) - J_{\text{near}}^\mp J_{\text{near}}^\pm$ takes the following simple form:

$$C_{\text{near}} = x^2 \partial_x^2. \quad (15)$$

The eigenvalue equations (9a) and (9b) near the boundary then become

$$i\partial_t \Phi_{\Delta,\omega}^{\text{near}}(t,x) = \omega \Phi_{\Delta,\omega}^{\text{near}}(t,x), \quad (16a)$$

$$\left(-\partial_x^2 + \frac{\Delta(\Delta-1)}{x^2}\right) \Phi_{\Delta,\omega}^{\text{near}}(t,x) = 0, \quad (16b)$$

which are easily solved with the result

$$\Phi_{\Delta,\omega}^{\text{near}}(t,x) = A_\Delta(\omega)x^\Delta e^{-i\omega t} + B_\Delta(\omega)x^{1-\Delta}e^{-i\omega t}, \quad (17)$$

where $A_\Delta(\omega)$ and $B_\Delta(\omega)$ are integration constants which may depend on Δ and ω . Substituting this into the ladder equations $J_{\text{near}}^\pm \Phi_{\Delta,\omega}^{\text{near}} \propto \Phi_{\Delta,\omega \pm i}^{\text{near}}$ we get

$$\begin{aligned} & (i\Delta \pm \omega)A_\Delta(\omega)x^\Delta e^{-i(\omega \pm i)t} + (i(1-\Delta) \pm \omega)B_\Delta(\omega)x^{1-\Delta}e^{-i(\omega \pm i)t} \\ & \propto A_\Delta(\omega \pm i)x^\Delta e^{-i(\omega \pm i)t} + B_\Delta(\omega \pm i)x^{1-\Delta}e^{-i(\omega \pm i)t}, \end{aligned} \quad (18)$$

which imply the following relations:

$$(i\Delta \pm \omega)A_\Delta(\omega) \propto A_\Delta(\omega \pm i), \quad (19a)$$

$$(i(1-\Delta) \pm \omega)B_\Delta(\omega) \propto B_\Delta(\omega \pm i). \quad (19b)$$

Now we are in a position to derive thermal two-point functions for a scalar primary operator of scaling dimension Δ . According to the real-time prescription of the AdS/CFT correspondence, the frequency-space two-point function for a scalar primary operator \mathcal{O}_Δ of scaling dimension Δ in the dual CFT₁ is given by the ratio [6, 7]

$$G_\Delta(\omega) = (2\Delta - 1) \frac{A_\Delta(\omega)}{B_\Delta(\omega)}. \quad (20)$$

It follows immediately from the relations (19a) and (19b) that the ratio (20) satisfies the recurrence relations

$$G_\Delta(\omega) = \frac{-1 + \Delta \pm i\omega}{-\Delta \pm i\omega} G_\Delta(\omega \pm i). \quad (21)$$

We emphasize that, in contrast to the Euclidean case [1], ω -dependence of $G_\Delta(\omega)$ is not uniquely determined from Eq. (21) and there are a number of nontrivial solutions that satisfy the recurrence relations. Among them are the following “minimal” solutions:²

$$G_\Delta^\pm(\omega) \propto e^{\pm\pi\omega} \Gamma(\Delta - i\omega) \Gamma(\Delta + i\omega), \quad (22a)$$

$$G_\Delta^{A/R}(\omega) \propto \frac{\Gamma(\Delta \pm i\omega)}{\Gamma(1 - \Delta \pm i\omega)}, \quad (22b)$$

where proportional coefficients are ω -independent. The solutions $G_\Delta^\pm(\omega)$ correspond to the Fourier transforms of positive- and negative-frequency two-point Wightman functions, $G_\Delta^+(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \mathcal{O}_\Delta(t) \mathcal{O}_\Delta(0) \rangle$ and $G_\Delta^-(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \mathcal{O}_\Delta(0) \mathcal{O}_\Delta(t) \rangle$,³ both of which have simple poles at $\omega = \pm i(\Delta + n)$ ($n = 0, 1, 2, \dots$). On the other hand, the solutions $G_\Delta^{A/R}(\omega)$ correspond to the Fourier transforms of advanced and retarded two-point functions, $G_\Delta^A(\omega) = i \int_{-\infty}^{\infty} dt \theta(-t) e^{i\omega t} \langle [\mathcal{O}_\Delta(t), \mathcal{O}_\Delta(0)] \rangle$ and $G_\Delta^R(\omega) = -i \int_{-\infty}^{\infty} dt \theta(t) e^{i\omega t} \langle [\mathcal{O}_\Delta(t), \mathcal{O}_\Delta(0)] \rangle$, each of which has simple poles only on the positive and negative imaginary ω -axis, $\omega = i(\Delta + n)$ and $\omega = -i(\Delta + n)$ ($n = 0, 1, 2, \dots$), respectively. It should be noted that the Fourier transforms of other real-time two-point functions such as the Pauli-Jordan commutator function $\langle [\mathcal{O}_\Delta(t), \mathcal{O}_\Delta(0)] \rangle$, the two-point Hadamard function $\langle \{\mathcal{O}_\Delta(t), \mathcal{O}_\Delta(0)\} \rangle$ and the Feynman propagator $\theta(t) \langle \mathcal{O}_\Delta(t) \mathcal{O}_\Delta(0) \rangle + \theta(-t) \langle \mathcal{O}_\Delta(0) \mathcal{O}_\Delta(t) \rangle$ are all given by appropriate linear combinations of (22a) and (22b).

²Here “minimal” means that $G_\Delta(\omega)$ does not contain a factor $f(\omega) = f(\omega \pm i)$ such as $e^{\pm 2n\pi\omega}$ ($n \in \mathbb{Z}$).

³ $\langle \mathcal{O}_\Delta(t_1) \mathcal{O}_\Delta(t_2) \rangle \propto \left[\frac{\pi T}{\sinh(\pi T(t_1 - t_2 - i\epsilon))} \right]^{2\Delta}$.

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